Challenges in Fiducial Inference

Parts of this talk are joint work with
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[a]NSF support acknowledged

Outline

- Introduction
- Definition
- Sparsity
- Regularization
- Conclusions

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Fiducial?

- Oxford English Dictionary
  - adjective technical (of a point or line) used as a fixed basis of comparison.
  - Origin from Latin fiducia ‘trust, confidence’
- Merriam-Webster dictionary
  1. taken as standard of reference *a fiducial mark*
  2. founded on faith or trust
  3. having the nature of a trust : fiduciary

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- Challenge of extra information:
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  - Regularization
- My point of view: frequentist
  - Justified using asymptotic theorems and simulations.
  - GFI tends to work well

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  - Likelihood as a distribution?

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$$\arg\min_{\xi} \|\mathbf{x} - \boldsymbol{G}(\mathbf{U}^{\star}, \xi)\| \mid \{\min_{\xi} \|\mathbf{x} - \boldsymbol{G}(\mathbf{U}^{\star}, \xi)\| \leq \varepsilon\} \quad (1)$$

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  - Similar to ABC; generating from prior replaced by $\min$.
  - Is this practicle? Can we compute?

Explicit limit (1)

- Assume $\mathbf{X} \in \mathbb{R}^n$ is continuous; parameter $\xi \in \mathbb{R}^p$
- The limit in (1) has density (H, Iyer, Lai & Lee, 2016)

$$r(\xi|\mathbf{x}) = \frac{f_{\mathbf{X}}(\mathbf{x}|\xi) J(\mathbf{x},\xi)}{\int_{\Xi} f_{\mathbf{X}}(\mathbf{x}|\xi') J(\mathbf{x},\xi')\, d\xi'},$$

where $J(\mathbf{x},\xi) = D\left(\frac{d}{d\xi} \mathbf{G}(\mathbf{u},\xi) \Big|_{\mathbf{u}=\mathbf{G}^{-1}(\mathbf{x},\xi)} \right)$

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- $n = p$ gives $D(A) = |\det A|$
- $\|\cdot\|_2$ gives $D(A) = (\det A^\top A)^{1/2}$
  Compare to Fraser, Reid, Marras & Yi (2010)
- $\|\cdot\|_\infty$ gives $D(A) = \sum_{\mathbf{i}=(i_1,\ldots,i_p)} |\det(A)_{\mathbf{i}}|$

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$$J(\boldsymbol{y}, \beta, \sigma) = \left(\det\left((\boldsymbol{X}, \frac{\boldsymbol{y} - \boldsymbol{X}\beta}{\sigma})^\top (\boldsymbol{X}, \frac{\boldsymbol{y} - \boldsymbol{X}\beta}{\sigma})\right)\right)^{1/2}$$

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- Fiducial *happens to be* same as independence Jeffreys, *explicit* normalizing constant

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  - In simulations fiducial was marginally better than reference prior which was much better than flat prior.

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- GFD is invariant to re-parametrizations (same as Jeffreys)
- GFD is *not* invariant to smooth transformation of the data if $n > p$
- Does not satisfy likelihood principle.

Various Asymptotic Results

$r(\xi|\mathbf{x}) \propto f_{\mathbf{X}}(\mathbf{x}|\xi) J(\mathbf{x},\xi)$ where $J(\mathbf{x},\xi) = D\left(\frac{d}{d\xi}\mathbf{G}(\mathbf{u},\xi)\Big|_{\mathbf{u}=\mathbf{G}^{-1}(\mathbf{x},\xi)} \right)$

- Most start with $C_n^{-1} J(\mathbf{x},\xi) \to J(\xi_0,\xi)$
- Bernstein-von Mises theorem for fiducial distributions provides asymptotic correctness of fiducial CIs H (2009, 2013), Sonderegger & H (2013) .
- Consistency of model selection H & Lee (2009), Lai, H & Lee (2015), H, Iyer, Lai & Lee (2016).
- Regular higher order asymptotics in Pal Majumdar & H (2016+).

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Theorem: (H, Iyer, Lai, Lee 2016) Under assumptions

$$r(M|\boldsymbol{y}) \propto q^{|M|} \int_{\boldsymbol{\xi}_M} f_M(\boldsymbol{y}, \boldsymbol{\xi}_M) J_M(\boldsymbol{y}, \boldsymbol{\xi}_M)\, d\boldsymbol{\xi}_M$$

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  - Default value $q = n^{-1/2}$ (motivated by MDL)

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$$r(M|\boldsymbol{y}) \propto \int_{\boldsymbol{\xi}_M} f_M(\boldsymbol{y}, \boldsymbol{\xi}_M) J_M(\boldsymbol{y}, \boldsymbol{\xi}_M) h_M(\boldsymbol{\xi}_M)\, d\boldsymbol{\xi}_M,$$

$$h_M(\boldsymbol{\xi}_M) = \begin{cases} 0 & \text{a smaller model predicts nearly as well} \\ 1 & \text{otherwise} \end{cases}$$

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- Motivated by non-local priors of Johnson & Rossell (2009)

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$$h_M(\beta_M) := I_{\left\{\frac{1}{2}\|X^T(X_M\beta_M - Xb_{min})\|_2^2 \geq \varepsilon(n, |M|)\right\}}$$

where b_{min} solves

$$\min_{b \in R^p} \frac{1}{2}\|X^T(X_M\beta_M - Xb)\|_2^2 \quad \text{subject to} \quad \|b\|_0 \leq |M| - 1.$$

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  - similar to Dantzig selector Candes & Tao (2007) different norm and target

GFD

$$r(M|\boldsymbol{y}) \propto \pi^{\frac{|M|}{2}} \Gamma\left(\frac{n-|M|}{2}\right) RSS_M^{-\left(\frac{n-|M|-1}{2}\right)} E[h_M^{\varepsilon}(\beta_M^{\star})]$$

Observations:

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- Implemented using Grouped Independence Metropolis Hastings (Andrieu & Roberts, 2009).

Main Result

Theorem Williams & H (2017+)
Suppose the true model is given by M_T. Then under certain conditions, for a fixed positive constant $\alpha < 1$,

$$r(M_T|y) = \frac{r(M_T|y)}{\sum_{j=1}^{n^\alpha} \sum_{M:|M|=j} r(M|y)} \xrightarrow{P} 1 \text{ as } n, p \to \infty.$$

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- For a large model $|M| > p_T$ and large enough n or p,

$$\frac{9}{2}\|X^T(H_M - H_{M(-1)})\mu_T\|_2^2 < \varepsilon_M(n,p),$$

where H_M and $H_{M(-1)}$ are the projection matrix for M and M with a covariate removed respectively.

Simulation

- Setup from Rockova & George (2015)
  - $n = 100$, $p = 1000$, $p_T = 8$.
  - Columns of X either a) independent or b) correlated with $\rho = 0.6$
  - $\varepsilon_M(n,p) = \Lambda_M \widehat{\sigma}^2_M \left(\frac{n^{0.51}}{9} + |M| \frac{\log(p\pi)^{1.1}}{9} - \log(n)^{\gamma} \right)_+$ with $\gamma = 1.45$.

Highlight of simulation results

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  - based on conditions: p decreased to 500 to satisfy, performance improves.

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- What if small model not the right target, e.g., gene interactions?

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Recall

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- Conditioning $U^{\star}$ on $\{\mathbf{x} = \boldsymbol{G}(\mathbf{U}^{\star}, \xi)\}$

  – “regularization by model”

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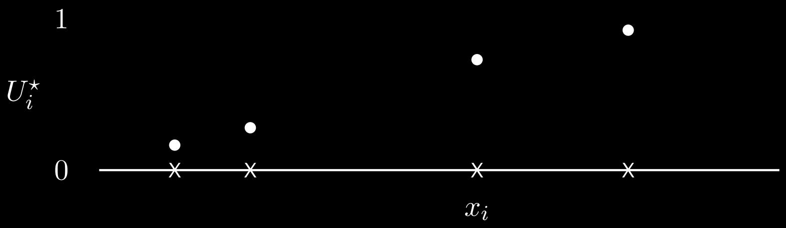

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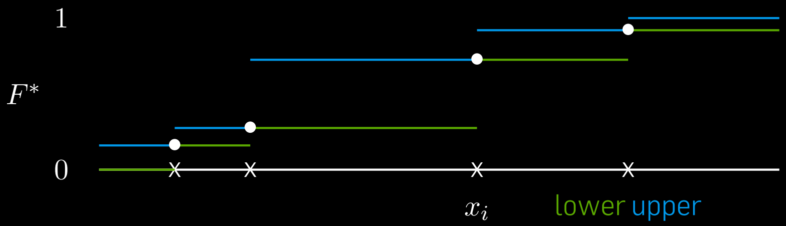

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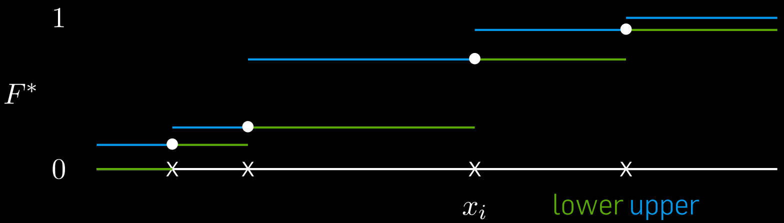

- See Yifan Cui's poster for extension to censored data.

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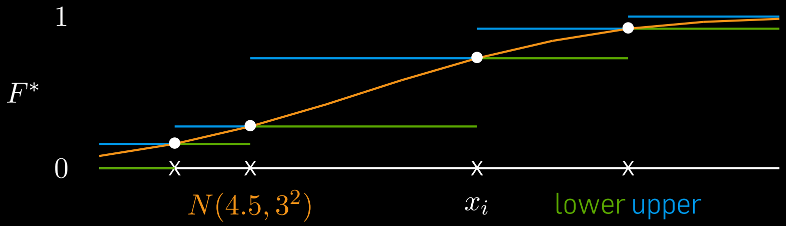

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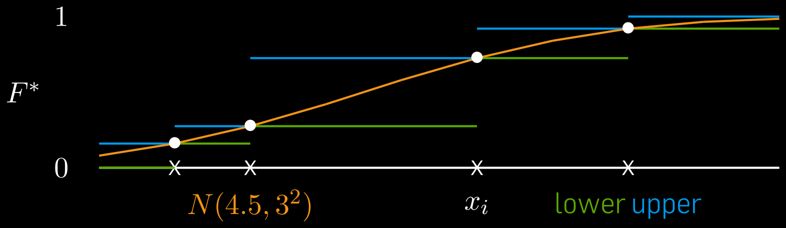

- GFD is $r(\mu, \sigma) \propto \sigma^{-1} \prod_{i=1}^n \sigma^{-1} f\left(\sigma^{-1}(x_i - \mu)\right)$

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- Option 3: mixture
  - $r(\mu) = \min(\frac{1}{2}, 1 - \Phi(x))) I_{\{0\}} + \max(\frac{1}{2\Phi(x)}, 1) \varphi(x - \mu) I_{\{\mu>0\}}$
  - Correct/conservative coverage, no $\{0\}$ for reasonable α CIs.

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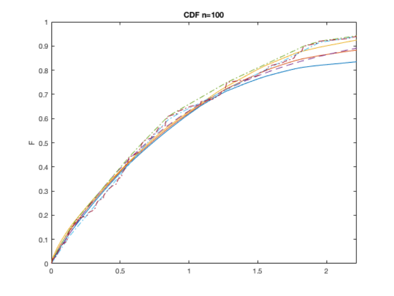
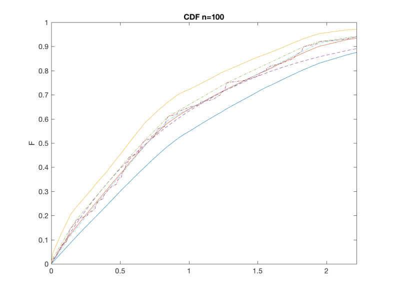


Condition



Projection

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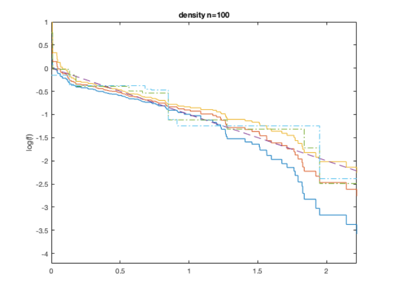

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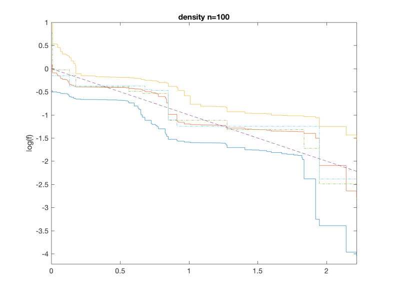

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Outline

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  - The proof is in the pudding

List of successful applications

- General Linear Mixed Models E, H & Iyer (2008); Cissewski & H (2012)
- Confidence sets for wavelet regression H & Lee (2009) and free knot splines Sonderegger & H (2014)
- Extreme value data (Generalized Pareto), Maximum mean, and model comparison Wandler & H (2011, 2012ab)
- Uncertainty quantification for ultra high dimensional regression Lai, H & Lee (2015), Wandler & H (2017+)
- Volatility estimation for high frequency data Katsoridas & H (2016+)
- Logistic regression with random effects (response models) Liu & H (2016,2017)

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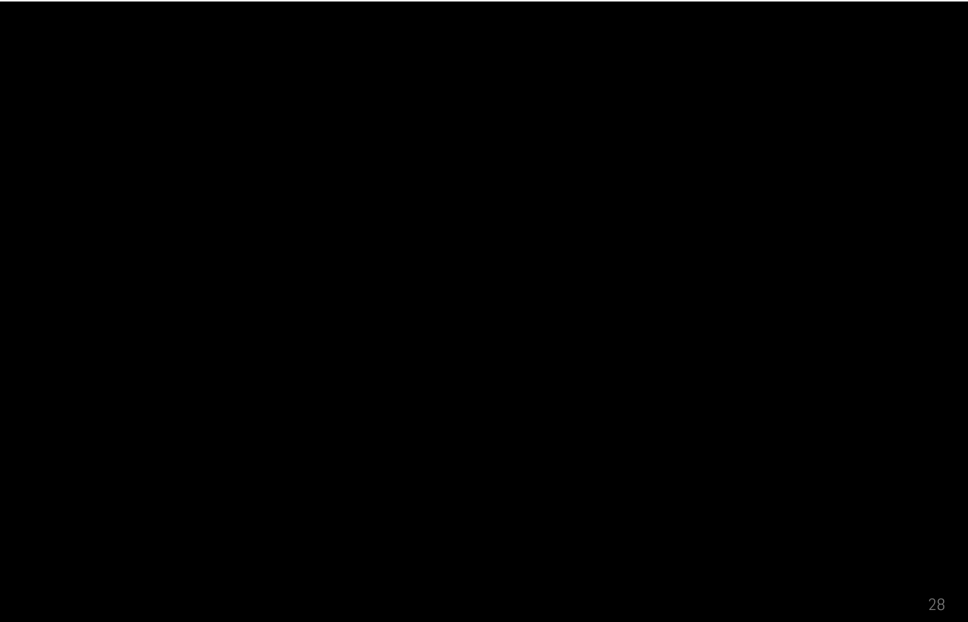

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Thank you!